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Extended-geostrophic Hamiltonian models for rotating shallow water motion

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Abstract

By using a small Rossby number expansion in Hamilton's principle for shallow water dynamics in a rapidly rotating reference frame, we derive new approximate extended-geostrophic equations with pressure, or surface height, as the dynamical variable. By introducing modeling parameters into the relation between the ageostrophic velocity and surface height, we obtain a family of models for which the functional form of the potential vorticity, conserved on fluid particles, and of the kinetic energy contribution to the globally conserved energy can be prescribed. The particular approximate equations which result when the ageostrophic velocity is chosen to be in full agreement with an asymptotic small Rossby number expansion of the rotating shallow water equations give a new model with apparently higher-order dynamical consistency than either quasigeostrophy or geostrophic momentum. The geostrophic momentum model is recovered as a special case among these new models.

Keywords: Balance; Geostrophic flow; Ocean; Atmosphere; Dynamics

1. Introduction

We are dealing with hierarchies of approximations obtained using Hamilton's principle (HP) for continuum fluid dynamics. The advantage of making approximations in HP for fluids is that the resulting Euler–Lagrange equations preserve certain types of structure – e.g., Hamiltonian structure and its implications such as energy conservation and Kelvin's circulation theorem – possessed by the unapproximated equations. Several types of approximations may be applied to HP. For example: (1) truncations of asymptotic expansions of the action using small dimensionless parameters, e.g., aspect ratio, Froude number or Rossby number; (2) averages of the action in time, space, phase, or some combination of these parameters; (3) substitution of solution ansatzes into HP which restrict the variations to a desired class. We will concentrate on the first type here, although any of these types of approximation may be applied to HP for the unapproximated equations before taking variations. The guidance for making the approximation in HP typically derives from applying asymptotic analysis to the unapproximated equations. However, such approximations may also arise as a version of "variation of parameters", when an assumed form of the desired solution is substituted

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into HP before variations are taken. The latter approach is used, e.g., in the method of "collective coordinates" and in the averaged Lagrangian method in physics [24].

In making approximations in HP for ideal fluid dynamics, the resulting approximate theory will automatically possess two essential properties that are fundamental: energy conservation and Kelvin's circulation theorem. By Stokes' theorem, the latter property leads to conservation of potential vorticity on fluid parcels. In the HP approach, these conservation laws are ensured by Noether's theorem, since they arise from symmetries of HP under, respectively, time translations and fluid parcel relabeling, each of which leaves invariant the Eulerian fluid velocity, free surface height, etc. Since the HP for ideal incompressible fluids is expressed only in terms of these Eulerian variables, it too is invariant. We emphasize at the outset that the HP approximation method we develop here applies in all of ideal continuum mechanics, including, e.g., plasma physics and nonlinear elasticity, as well as fluids. In fact, the approach applies to any dynamical equations derivable from HP.

Here, we use the HP approximation method to develop new models of rotating shallow water (RSW) dynamics for small Rossby number, denoted $\epsilon \ll 1$. The models we develop are extended-geostrophic (EG) approximations, which are intermediate in accuracy between the quasigeostrophic (QG) description and the full equations for RSW dynamics with a free surface moving under gravity. The asymptotic QG theory is used as a guide for making new EG approximations in HP for the RSW theory. Truncating the expansion of HP for RSW dynamics leads successively to geostrophic balance at leading order; Salmon's HP model [19] at order $O(\epsilon)$; and to a family of EG models, dependent on the choice of modeling parameters, at order $O(\epsilon^2)$. In the latter case, we introduce four modeling parameters: τ, which allows for different approximations of the potential vorticity; α, which provides a relative weighting among different parts of the ageostrophic velocity; γ, which allows different approximations for the conserved energy; and β, which allows weak spatial variations in the Coriolis parameter. The geostrophic momentum (GM) equations [14] are recovered for particular values of the modeling parameters.

The primary motivation for this study follows from results of numerical experiments [2,3,5] demonstrating clearly that at moderate values of ϵ, the GM model [14] and Salmon's HP model [19] provide disappointingly inaccurate approximate solutions to the RSW equations compared, e.g., to these obtained from the balance equations (BE). This is in spite of the fact that GM and Salmon's HP model have Hamiltonian structure and prossess conservation laws for global energy and for potential vorticity on fluid particles, whereas the BE for the RSW equations do not conserve energy. Thus, possession of Hamiltonian structure is not sufficient in itself to ensure an accurate approximate model. The question then arises of whether a more accurate approximate model with Hamiltonian structure can be formulated. We show here that a family of approximate EG models for RSW dynamics may be derived from HP by using a small Rossby number ϵ-expansion in the RSW equations and by adding modeling parameters to vary retained terms. The GM equations and Salmon's HP model are recovered for certain values of the modeling parameters. One particular new EG model is distinguished when the modeling parameters are chosen to be those dictated by an asymptotic expansion of the RSW equations to $O(\epsilon)$. Consequently, this model may be expected to provide approximate solutions to the RSW equations with greater accuracy than those obtained with GM or Salmon's HP model. Numerical studies designed to test this expectation will be reported elsewhere.

Bottom topography and spatial variation of the Coriolis parameter are easily incorporated into HP for fluids when variations of Lagrangian labels are taken at fixed Eulerian position and time. An alternative technically equivalent approach is to take variations of particle paths at constant Lagrangian label, but this is less convenient when the bottom topography and Coriolis parameter vary with Eulerian position. Salmon [19–22] discusses HP for shallow water dynamics using such variations of particle paths at constant Lagrangian label, with applications to semigeostrophic models in [22]. Abarbanel and Holm [1] and Holm [12] discuss HP for ideal incompressible fluid dynamics in three dimensions using both these alternative approaches, as well as discussing the use of the resulting Hamiltonian formulation in establishing Lyapunov stability conditions for equilibrium solutions. Holm et al. [11] discuss these two alternative approaches from the Hamiltonian viewpoint and show that the transformation from the

Lagrangian description of ideal continuum mechanics to the Eulerian description is a Poisson map (i.e., preserves Poisson brackets). We shall use this result in Section 2.

In the rotating shallow water case, our strategy is to expand HP for shallow water dynamics around leading order geostrophic balance in powers of the Rossby number, ϵ, which we assume is small ($\epsilon \ll 1$). For simplicity, we will assume that the variations in the Coriolis parameter and in the bottom topography are $O(\epsilon)$. At each order in ϵ, an expression for the momentum density as a function of the other Eulerian fluid variables appears as a *constraint*, which is imposed by using the fluid velocity $\boldsymbol{u}$ as a Lagrange multiplier. The leading order terms in HP for RSW dynamics yield geostrophic balance. The order $O(\epsilon)$ terms incorporate into HP the kinetic energy density and momentum density due to geostrophic horizontal motion. Truncation at this order in HP gives Salmon's model [19]. The fluid theory we derive at order $O(\epsilon^2)$ in HP results in a family of models that depend on the choice of values for the modeling parameters τ, α, γ, and β. By construction, the EG fluid theory obtained from the expansion of HP for RSW at each order in the Rossby number expansion constrain yet preserve the RSW Hamiltonian structure. So that theory at each order conserves global energy and possesses a Kelvin's theorem which implies conservation of potential vorticity on fluid parcels.

The plan of the paper is as follows. Section 2 develops the theory of HP for fluid motion in the Eulerian representation so as to allow singular Lagrangians which are linear in the fluid velocity. This theory possesses a modeling freedom that allows one to choose in the modeling procedure *arbitrary* forms of the momentum density and energy density as *independent* functions of the free surface height and its spatial derivatives. In Section 3 we use this freedom to develop HP for EG Eulerian shallow water models at orders $O(\epsilon)$ and $O(\epsilon^2)$ in the expansion of the HP for RSW dynamics in Rossby number, ϵ. We use QG theory as a guide in posing the model equation (3.20) for the ageostrophic fluid velocity. The order $O(\epsilon^2)$ model reduces to the GM equations for a certain choice of the modeling parameters. Otherwise, the model conserves energy and has a Kelvin circulation theorem which depends on the choice of modeling parameters. The effect of varying these modeling parameters in numerical solutions of these equations will be studied elsewhere.

2. Hamilton's principle for fluid motion in two dimensions

2.1. The general case

2.1.1. HP for variations in Lagrangian fluid labels

HP determines the dynamics of a continuum fluid medium moving in two space dimensions by requiring stationarity of the action, L, given by

$$L = \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \mathcal{L}(l^A_{,t}, \nabla l^A, \nabla\nabla l^A, \text{etc.}), \tag{2.1}$$

under variations of Lagrangian labels $l^A(\boldsymbol{x}, t)$, $A = 1, 2$, at fixed Eulerian position $\boldsymbol{x} = (x, y)$ and time t. The Lagrangian label $l^A(\boldsymbol{x}, t)$ moves with the fluid, so it satisfies the following characteristic equations:

$$\frac{\mathrm{d}l^A}{\mathrm{d}t} \equiv l^A_{,t} + u^i l^A_{,i} = 0, \quad i, A = 1, 2, \tag{2.2}$$

in which we sum over repeated indices and use subscript notation for partial derivatives. Consequently, the deformation gradient $D^A_i = l^A_{,i}$ satisfies

$$\partial_t D^A_i + \boldsymbol{u} \cdot \nabla D^A_i + D^A_j u^j_{,i} = 0, \quad i, A = 1, 2, \tag{2.3}$$

and its determinant $D = \det(D_i^A)$ obeys the continuity equation,

$$D_{,t} + \nabla \cdot D\boldsymbol{u} = 0. \tag{2.4}$$

The dependent variables for incompressible shallow water motion are the Eulerian fluid velocity $\boldsymbol{u}$ and the total depth of the water η. These are given in terms of derivatives of the Lagrangian fluid labels by (cf. Eq. (2.2))

$$u^i = -(D^{-1})^i_A l^A_{,t} \quad \text{and} \quad \eta = D. \tag{2.5}$$

That is, three dimensional incompressiblity implies the fluid depth η is given by the Jacobian for the transformation from the current Eulerian position $\boldsymbol{x}$ to the initial Lagrangian label l^A with $A = 1, 2$. See, e.g., [12] for more details.

To develop the theory of approximate continuum dynamics models based on HP, we first rewrite the action in Eq. (2.1) as

$$L = \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \left[l^A_{,t} \frac{\partial \mathcal{L}}{\partial l^A_{,t}} - \overline{\mathcal{H}}(l^A_{,t}, \nabla l^A, \nabla\nabla l^A, \text{etc.}) \right]. \tag{2.6}$$

The Hamiltonian $H(\pi_A, l^A)$, with canonical momentum density defined by $\pi_A \equiv \delta L/\delta l^A_{,t}$ is found from the Legendre transform of the action $L(l^A_{,t}, l^A)$. Namely,

$$H = \int \mathrm{d}x\,\mathrm{d}y \left(\pi_A - \frac{\partial \mathcal{L}}{\partial l^A_{,t}} \right) l^A_{,t} + \overline{\mathcal{H}}(l^A_{,t} \nabla l^A, \nabla\nabla l^A, \text{etc.}). \tag{2.7}$$

Hamilton's canonical equations are:

$$\begin{aligned} 0 &= \frac{\delta H}{\delta l^A_{,t}} = \pi_A - \frac{\partial^2 \mathcal{L}}{\partial l^A_{,t} \partial l^B_{,t}} l^B_{,t} - \frac{\partial \mathcal{L}}{\partial l^A_{,t}} + \frac{\partial \overline{\mathcal{H}}}{\partial l^A_{,t}}, \\ l^A_{,t} &= \frac{\delta H}{\delta \pi_A} = \{l^A, H\}_{\text{can}} = -u^i l^A_{,i}, \qquad \pi_{A,t} = -\frac{\delta H}{\delta l^A} = \{\pi_A, H\}_{\text{can}}, \end{aligned} \tag{2.8}$$

where $\{\cdot, \cdot\}_{\text{can}}$ denotes the canonical Poisson bracket,

$$\{F, H\}_{\text{can}} = \int \mathrm{d}x\,\mathrm{d}y \left(\frac{\delta F}{\delta l^A} \frac{\delta H}{\delta \pi_A} - \frac{\delta H}{\delta l^A} \frac{\delta F}{\delta \pi_A} \right). \tag{2.9}$$

The first equation in (2.8) relates the canonical momentum density π_A in phase space to the configuration space variables l^A and $l^A_{,t}$. Of course, this relation is equivalent to the usual one. For Lagrangians which are purely quadratic in $l^A_{,t}$, the last two terms cancel in the relation for π_A in (2.8) and solvability of the relation for the canonical momentum density as a function of the configuration space variables depends on whether the matrix of second derivatives appearing in that relation is invertible. For Lagrangians that are linear in velocity, however, that matrix vanishes and both the quantity $\partial \mathcal{L}/\partial l^A_{,t}$ and the energy density $\overline{\mathcal{H}}$ in Eq. (2.7) are functions of the spatial derivatives of l^A only. Hence, the quantity $\partial \overline{\mathcal{H}}/\partial l^A_{,t}$ in (2.8) also vanishes. In this case, requiring that the defining relation for the canonical momentum density,

$$\pi_A = \overline{\pi}_A(\nabla l^A) = \partial \mathcal{L}/\partial l^A_{,t}, \tag{2.10}$$

be preserved under the ensuing Hamiltonian dynamics leads to equations for the Lagrange multipliers $l^A_{,t}$ in (2.7) which enforce the defining relation (2.10) for the momentum density as a function of the spatial derivatives of the Lagrangian fluid labels.

2.1.2. Remark on Lagrangian submanifolds

In finite dimensions, a relation such as (2.10) between canonical momentum p and coordinate q (e.g., $p = \partial S(q)/\partial q$) would define a Lagrangian submanifold of the corresponding phase space (p, q). This is a manifold whose dimension is equal to the dimension of the configuration space and on which the canonical two form $\mathrm{d}p \wedge \mathrm{d}q$ defining the symplectic structure on the phase space is identically zero. Lagrangian submanifolds arise, for example, in geometrical ray optics, see, e.g., [4]. Lagrangian submanifolds arise in making EG approximations in shallow water dynamics because EG theory deals with singular Lagrangians.

2.1.3. Transformation to Eulerian fluid variables

We define the Eulerian momentum density $\boldsymbol{m}$ to be

$$\boldsymbol{m} = \delta L/\delta \boldsymbol{u} = -\pi_A \nabla l^A \tag{2.11}$$

for which Eq. (2.5) implies the useful relation

$$\pi_A l^A_{,t} = \boldsymbol{m} \cdot \boldsymbol{u}. \tag{2.12}$$

Hence, we may transform the Hamiltonian in (2.7) to

$$H = \int \mathrm{d}x\, \mathrm{d}y \left(\boldsymbol{m} - \frac{\partial L}{\partial \boldsymbol{u}}\right) \cdot \boldsymbol{u} + \overline{\mathcal{H}}(\boldsymbol{u}, \nabla l^A, \nabla\nabla l^A, \text{etc.}). \tag{2.13}$$

For isotropic fluids, the dependence in $\overline{\mathcal{H}}$ on ∇l^A is only though the fluid depth, η. So we may write the Hamiltonian for isotropic fluids as

$$H(\boldsymbol{m}, \eta) = \int \mathrm{d}x\, \mathrm{d}y \left(\boldsymbol{m} - \frac{\partial L}{\partial \boldsymbol{u}}\right) \cdot \boldsymbol{u} + \overline{\mathcal{H}}(\boldsymbol{u}, \eta, \nabla\eta, \text{etc.}). \tag{2.14}$$

Transforming Hamilton's canonical equations (2.8) to the Eulerian fluid variables, $\boldsymbol{m}$, $\boldsymbol{u}$ and η, gives the following relations:

$$\begin{aligned}
0 &= \frac{\delta H}{\delta \boldsymbol{u}} = \boldsymbol{m} - \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{u} \partial \boldsymbol{u}} \cdot \boldsymbol{u} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} + \frac{\partial \overline{\mathcal{H}}}{\partial \boldsymbol{u}}, \\
\boldsymbol{u} &= \frac{\delta H}{\delta \boldsymbol{m}}, \\
\frac{\partial \eta}{\partial t} &= \{\eta, H\} \equiv -\partial_j \eta \frac{\delta H}{\delta m_j} = -\nabla \cdot \eta \boldsymbol{u}, \\
\frac{\partial m_i}{\partial t} &= \{m_i, H\} \equiv -(\partial_j m_i + m_j \partial_i) \frac{\delta H}{\delta m_j} - \eta \partial_i \frac{\delta H}{\delta \eta},
\end{aligned} \tag{2.15}$$

where $\partial_i = \partial x^i$, $i = 1, 2$, acts on all terms standing to its right in a product. The first equation in (2.15) relates the Eulerian momentum density $\boldsymbol{m}$ to other Eulerian fluid variables, $\boldsymbol{u}$, η, and (possibly) $\nabla\eta$. All the equations in (2.15) are variable transformations of the momentum density relation and Hamilton's canonical equations in the set (2.8).

2.1.4. Remark on Lie–Poisson brackets

The bracket notation in Eq. (2.15) denotes the Lie–Poisson bracket

$$\{F, H\}(\boldsymbol{m}, \eta) = -\int \mathrm{d}x\, \mathrm{d}y \left\{ \frac{\delta F}{\delta m_i} \left[(\partial_j m_i + m_j \partial_i) \frac{\delta H}{\delta m_j} + \eta \partial_i \frac{\delta H}{\delta \eta} \right] + \frac{\delta F}{\delta \eta} \partial_j \eta \frac{\delta H}{\delta m_j} \right\}. \tag{2.16}$$

This Lie–Poisson bracket was obtained in [11] from the canonical Poisson bracket in (2.9), by simply applying the chain rule for the change of variables from Lagrangian to Eulerian quantities,

$$(\pi_A, l^A) \to (\boldsymbol{m}, \eta) = (-\pi_A \nabla l^A, \det \nabla l^A), \tag{2.17}$$

and proving that this change of variables is a Poisson map (i.e., it preserves Poisson brackets). For more discussion of the role of Poisson maps in continuum mechanics and the mathematical properties of Lie–Poisson brackets, see, e.g., [10–13], and references therein. For a review of Hamiltonian fluid mechanics, see [21].

2.1.5. The classical relations of ideal fluid dynamics

The last three equations in (2.15) combine to yield the equation of fluid motion in either the Lie-derivative form,

$$\frac{\partial}{\partial t}\left(\frac{1}{\eta}\boldsymbol{m}\right) + (\boldsymbol{u}\cdot\nabla)\frac{1}{\eta}\boldsymbol{m} + \frac{1}{\eta}m_j \nabla u^j + \nabla\frac{\delta H}{\delta \eta} = 0, \tag{2.18}$$

or, in the equivalent curl form,

$$\frac{\partial}{\partial t}\left(\frac{1}{\eta}\boldsymbol{m}\right) + \left(\operatorname{curl}\frac{1}{\eta}\boldsymbol{m}\right) \times \boldsymbol{u} + \nabla\left(\frac{\delta H}{\delta \eta} + \frac{1}{\eta}\boldsymbol{m}\cdot\boldsymbol{u}\right) = 0, \tag{2.19}$$

obtained from Eq. (2.18) by using the fundamental vector identity of fluid dynamics,

$$(\operatorname{curl}\boldsymbol{a}) \times \boldsymbol{b} + \nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + a_j \nabla b^j, \tag{2.20}$$

with, in this case, $\boldsymbol{a} = \boldsymbol{m}/\eta$ and $\boldsymbol{b} = \boldsymbol{u}$. The Lie-derivative form (2.18) results naturally in Kelvin's circulation theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \frac{1}{\eta}\boldsymbol{m}\cdot\mathbf{d}\boldsymbol{x} = \oint_{\gamma(t)} \left[\frac{\partial}{\partial t}\left(\frac{1}{\eta}\boldsymbol{m}\right) + (\boldsymbol{u}\cdot\nabla)\frac{1}{\eta}\boldsymbol{m} + \frac{1}{\eta}m_j\nabla u^j\right]\cdot\mathbf{d}\boldsymbol{x} = -\oint_{\gamma(t)} \nabla\frac{\delta H}{\delta\eta}\cdot\mathbf{d}\boldsymbol{x} = 0, \tag{2.21}$$

for any closed loop $\gamma(t)$ moving with the fluid. This is the fluid analog of invariance of the Poincaré action integral $\oint p\,\mathrm{d}q$ in classical mechanics, since by Eq. (2.17)

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \frac{1}{\eta}\boldsymbol{m}\cdot\mathbf{d}\boldsymbol{x} = -\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \frac{\pi_A}{\eta}\,\mathrm{d}l^A = 0. \tag{2.22}$$

Taking the vector product of $\hat{z}$ with the curl form of the fluid motion equation (2.19) gives an expression for the fluid velocity in a "Bernoulli relation",

$$\eta Q\boldsymbol{u} = \hat{z}\times\left(\nabla B + \frac{\partial}{\partial t}\left(\frac{1}{\eta}\boldsymbol{m}\right)\right), \tag{2.23}$$

where Q is the potential vorticity,

$$Q \equiv \frac{1}{\eta}\hat{z}\cdot\left(\operatorname{curl}\frac{1}{\eta}\boldsymbol{m}\right), \tag{2.24}$$

and B is the Bernoulli function, defined by

$$B = \frac{\delta H}{\delta\eta} + \frac{1}{\eta}\boldsymbol{m}\cdot\boldsymbol{u}. \tag{2.25}$$

Either applying Stokes's theorem in the motion equation in its Kelvin theorem form (2.21), or taking its curl in the form (2.19), immediately implies the advection law,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \equiv \frac{\partial Q}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} Q = 0, \tag{2.26}$$

for the potential vorticity Q in (2.24).

2.1.6. *The momentum constraint*

The Eulerian momentum density $\boldsymbol{m}$ is given in terms of the other Eulerian fluid variables by the condition $\delta H/\delta \boldsymbol{u} = 0$ in Eq. (2.15). Thus, in this formulation the fluid velocity $\boldsymbol{u}$ in the Hamiltonian (2.13) appears as a vector Lagrange multiplier which enforces the relation of the Eulerian momentum density to the other Eulerian fluid variables as a dynamically preserved constraint. This definition is usually taken for granted. However, in what follows we shall *model* the momentum density as a prescribed function of the other fluid variables, for example,

$$\boldsymbol{m} = \overline{\boldsymbol{m}}(\eta, \boldsymbol{\nabla}\eta, \text{etc.}) \equiv \overline{\boldsymbol{m}}[\eta], \tag{2.27}$$

where $\overline{\boldsymbol{m}}[\eta]$ may be a functional of η and its spatial derivatives. In this type of modeling, we shall need the explicit enforcement of the momentum definition (2.27), both as a constraint and as a means of determining the fluid velocity for the model flow by using the method of Lagrange multipliers.

2.1.7. *Linear fluid Lagrangians*

For the case (2.27), the action in (2.6) is a *linear* function of the fluid velocity, $\boldsymbol{u}$,

$$L = \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \overline{\boldsymbol{m}}[\eta] \cdot \boldsymbol{u} - \overline{\mathcal{H}}[\eta] \tag{2.28}$$

with potentially arbitrary forms of $\overline{\boldsymbol{m}}[\eta]$ and $\overline{\mathcal{H}}[\eta]$. From (2.14), the corresponding Hamiltonian is then

$$H = \int \mathrm{d}x\, \mathrm{d}y (\boldsymbol{m} - \overline{\boldsymbol{m}}[\eta]) \cdot \boldsymbol{u} + \overline{\mathcal{H}}[\eta], \tag{2.29}$$

whose variational derivatives are given by

$$\delta H = \int \mathrm{d}x\, \mathrm{d}y \left[\boldsymbol{u} \cdot \delta \boldsymbol{m} + (\boldsymbol{m} - \overline{\boldsymbol{m}}[\eta]) \cdot \delta \boldsymbol{u} + \left(\frac{\delta H}{\delta \eta} \right) \delta \eta \right] \tag{2.30}$$

with

$$\frac{\delta H}{\delta \eta} = \frac{\delta \overline{\mathcal{H}}}{\delta \eta} - \frac{\delta \overline{\boldsymbol{m}}}{\delta \eta} \cdot \boldsymbol{u}. \tag{2.31}$$

For each modeling choice of the functions $\overline{\boldsymbol{m}}[\eta]$ and $\overline{\mathcal{H}}[\eta]$, the fluid velocity $\boldsymbol{u}$ must be determined by analyzing the motion equation (cf. Eq. (2.19))

$$\frac{\partial}{\partial t} \left(\frac{1}{\eta} \overline{\boldsymbol{m}}[\eta] \right) + \left(\mathrm{curl} \frac{1}{\eta} \overline{\boldsymbol{m}}[\eta] \right) \times \boldsymbol{u} + \boldsymbol{\nabla} \left[\frac{\delta H}{\delta \eta}[\eta] + \boldsymbol{u} \cdot \frac{1}{\eta} \overline{\boldsymbol{m}}[\eta] \right] = 0, \tag{2.32}$$

and requiring it to be compatible with the continuity equation for η,

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot \eta \boldsymbol{u} = 0. \tag{2.33}$$

Because of the forms of the motion equation (2.32) and the continuity equation (2.33), the classical relations of ideal fluid dynamics (2.23)–(2.26) are retained in the constrained (barred) variables, as well, even though the theory now possesses only one dynamical degree of freedom – the surface height. However, these relations only have meaning so long as compatibility of the system (2.32) and (2.33) provides a solvable equation for $\boldsymbol{u}$, as we shall assume here. A detailed discussion of the effects of loss of solvability for $\boldsymbol{u}$ in this system is beyond the scope of the present work, although this is clearly a useful endeavor for later work.

2.1.8. *Conservation laws*

The freedom to choose the functional forms of $\overline{\boldsymbol{m}}[\eta]$ and $\overline{\mathcal{H}}[\eta]$ in (2.29) opens a variety of modeling possibilities, as we will illustrate in later sections for EG flows. For each modeling choice, the structure of the theory ensures that the resulting approximate equations (2.32) and (2.33) conserve the integrated energy, given by

$$E = \int \mathrm{d}x\,\mathrm{d}y\overline{\mathcal{H}}[\eta], \tag{2.34}$$

and conserve the potential vorticity, on fluid parcels, i.e.,

$$\frac{\partial \overline{Q}}{\partial t} + \boldsymbol{u} \cdot \nabla \overline{Q} = 0, \tag{2.35}$$

where potential vorticity is defined as

$$\overline{Q} \equiv \frac{1}{\eta}\hat{z} \cdot \operatorname{curl}\frac{1}{\eta}\overline{\boldsymbol{m}}[\eta]. \tag{2.36}$$

Consequently, the integral quantity (Casimir function [13]),

$$C_\Phi = \int \mathrm{d}x\,\mathrm{d}y\eta\Phi(\overline{Q}), \tag{2.37}$$

is invariant under the model flow, for any function Φ, and for *any* choice of the functions $\overline{\boldsymbol{m}}[\eta]$ and $\overline{\mathcal{H}}[\eta]$ in the model Hamiltonian (so long as the system (2.32) and (2.33) remains solvable). The freedom to choose the functions $\overline{\boldsymbol{m}}[\eta]$ and $\overline{\mathcal{H}}[\eta]$ is illustrated in our development of EG shallow water models in Section 2.2. Before proceeding to develop these EG models in two dimensions, we note that the foregoing theory may be generalized to apply with appropriate modifications in any number of dimensions.

2.2. *Shallow water example*

To explain ideas and notation, we begin in the context of a familiar example. Namely, we discuss HP for the dimensionless RSW equations with variable Coriolis parameter $f = f(\boldsymbol{x})$ and bottom topography $b = b(\boldsymbol{x})$. These equations are

$$\epsilon\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + \nabla\left(\frac{\eta - b}{\epsilon\mathcal{F}}\right) = 0, \qquad \frac{\partial \eta}{\partial t} + \nabla \cdot \eta\boldsymbol{u} = 0. \tag{2.38}$$

The dimensional scales $(b_0, L, \mathcal{U}_0, f_0, g)$ in RSW dynamics denote characteristic fluid depth, horizontal length, horizontal fluid velocity, reference Coriolis parameter, and gravitational acceleration, respectively. Dimensionless quantities in Eq. (2.38) are unadorned and are related to their dimensional counterparts (decorated with primes), according to

$$\begin{aligned} &\boldsymbol{u}' = \mathcal{U}_0\boldsymbol{u}, \qquad \boldsymbol{x}' = L\boldsymbol{x}, \qquad t' = (L/\mathcal{U}_0)t, \qquad f' = f_0 f, \\ &b' = b_0 b, \qquad \eta' = b_0\eta, \qquad \eta' - b' = b_0(\eta - b). \end{aligned} \tag{2.39}$$

The dimensional quantities are: $\boldsymbol{u}'$, the horizontal fluid velocity; η', the fluid depth; b', the equilibrium depth; and $\eta' - b'$, the free surface elevation.

The dimensionless quantities ϵ and $\mathcal{F}$ appearing in the RSW equations (2.38) are the Rossby number and the squared ratio of the typical horizontal scale L to the external Rossby deformation radius, L_{R}, respectively. These quantities are given by

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L} \quad \text{and} \quad \mathcal{F} = \frac{L^2}{L_{\mathrm{R}}^2} \quad \text{with } L_{\mathrm{R}}^2 = \frac{g b_0}{f_0^2}. \tag{2.40}$$

For barotropic horizontal motions at length scales L in the ocean for which the squared external Rossby ratio $\mathcal{F}$ is of order O(1) – as we shall assume – the Rossby number ϵ is typically quite small ($\epsilon \ll 1$) and, thus, is a natural parameter for making asymptotic expansions.

The RSW equation (2.38) follow from HP, namely $\delta L_{\mathrm{RSW}} = 0$, with action L_{RSW} given (with $\operatorname{curl} \boldsymbol{R} = f(\boldsymbol{x})\hat{z}$) by

$$L_{\mathrm{RSW}} = \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \left[\eta \boldsymbol{u} \cdot \boldsymbol{R} - \frac{(\eta - b)^2}{2\epsilon \mathcal{F}} + \frac{\epsilon}{2} \eta |\boldsymbol{u}|^2 \right]. \tag{2.41}$$

We rearrange the action L_{RSW} as

$$L_{\mathrm{RSW}} = \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \left[\overline{\boldsymbol{m}} \cdot \boldsymbol{u} - \frac{(\eta - b^2)}{2\epsilon \mathcal{F}} - \frac{\epsilon}{2} \eta |\boldsymbol{u}|^2 \right]. \tag{2.42}$$

in which the Eulerian momentum density is given by

$$\overline{\boldsymbol{m}} \equiv \frac{\delta L_{\mathrm{RSW}}}{\delta \boldsymbol{u}} = \eta \boldsymbol{R} + \epsilon \eta \boldsymbol{u}. \tag{2.43}$$

Legendre transforming L_{RSW} gives the constrained RSW Hamiltonian (cf. Eq. (2.14)),

$$H_{\mathrm{RSW}} = \int \mathrm{d}x\,\mathrm{d}y \left[(\boldsymbol{m} - \overline{\boldsymbol{m}}) \cdot \boldsymbol{u} + \frac{(\eta - b)^2}{2\epsilon \mathcal{F}} + \frac{\epsilon}{2} \eta |\boldsymbol{u}|^2 \right], \tag{2.44}$$

whose variational derivatives are given by (cf. Eq. (2.30))

$$\delta H_{\mathrm{RSW}} = \int \mathrm{d}x\,\mathrm{d}y \left[\boldsymbol{u} \cdot \delta \boldsymbol{m} + (\boldsymbol{m} - \epsilon \eta \boldsymbol{u} - \eta \boldsymbol{R}) \cdot \delta \boldsymbol{u} + \left(\frac{\eta - b}{\epsilon \mathcal{F}} - \frac{\epsilon}{2} |\boldsymbol{u}|^2 - \boldsymbol{u} \cdot \boldsymbol{R} \right) \delta \eta \right]. \tag{2.45}$$

In this case, the velocity $\boldsymbol{u}$ is easily obtained in terms of the momentum density $\boldsymbol{m}$ and fluid depth η as

$$\boldsymbol{u} = (\boldsymbol{m} - \eta \boldsymbol{R})/\epsilon \eta \tag{2.46}$$

from the constraint equation, $\delta H_{\mathrm{RSW}}/\delta \boldsymbol{u} = 0$. Upon evaluating $\boldsymbol{m}$ in terms of $\boldsymbol{u}$ and η, and calculating $\delta H_{\mathrm{RSW}}/\delta \eta$ by using (2.45), the motion equation (2.18), or equivalently, (2.19) appears in the form

$$\epsilon \frac{\partial \boldsymbol{u}}{\partial t} + (f\hat{z} + \epsilon \operatorname{curl} \boldsymbol{u}) \times \boldsymbol{u} + \nabla \left(\frac{\eta - b}{\epsilon \mathcal{F}} + \frac{\epsilon}{2} |\boldsymbol{u}|^2 \right) = 0. \tag{2.47}$$

This transforms easily to the RSW motion equation in (2.38) upon using the identity (2.20).

2.3. *Quasigeostrophic approximation*

In this section, we cast the well-known QG approximation [16] of the equations for RSW motion in a rotating frame into a form which will be useful for formulating the Hamiltonian EG models developed later. Consistent with the QG approximation, we assume $f(\boldsymbol{x}) = 1 + \epsilon f_1(\boldsymbol{x})$ and $b(\boldsymbol{x}) = 1 + \epsilon b_1(\boldsymbol{x})$. We return to the RSW motion equation in (2.38), rewritten as

$$\epsilon \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -f\hat{z} \times \boldsymbol{u} - \boldsymbol{\nabla} h, \tag{2.48}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}, \qquad h = \frac{\eta - b}{\epsilon \mathcal{F}}. \tag{2.49}$$

Operating with $\hat{z}\times$ on Eq. (2.48) and expanding in powers of ϵ yields

$$\boldsymbol{u} = \hat{z} \times \boldsymbol{\nabla} h - \epsilon f_1 \hat{z} \times \boldsymbol{\nabla} h - \epsilon \left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} \right) \boldsymbol{\nabla} h + \mathrm{O}(\epsilon^2) = \boldsymbol{u}_{\mathrm{G}} + \epsilon \boldsymbol{u}_{\mathrm{A}} + \mathrm{O}(\epsilon^2), \tag{2.50}$$

where the geostrophic and ageostrophic components of the velocity are defined, respectively, by

$$\boldsymbol{u}_{\mathrm{G}} = \hat{z} \times \boldsymbol{\nabla} h \quad \text{and} \quad \boldsymbol{u}_{\mathrm{A}} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} \right) \hat{z} \times \boldsymbol{u}_{\mathrm{G}} - f_1 \boldsymbol{u}_{\mathrm{G}}. \tag{2.51}$$

The remainder of this paper is devoted to studying the class of RSW flows that satisfy condition (2.50). In Eq. (2.51), $\boldsymbol{u}_{\mathrm{G}}$ is divergenceless and $\boldsymbol{u}_{\mathrm{A}}$ has divergence given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{A}} = -\left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} \right) \nabla^2 h - \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} f_1, \tag{2.52}$$

in which ∇^2 is the horizontal Laplacian. Substituting expression (2.52) for $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{A}}$ into the continuity equation (2.33) rewritten as $\epsilon \mathcal{F} h_{,t} = -\boldsymbol{\nabla} \cdot \eta \boldsymbol{u}$ and using the relations $\eta = b + \epsilon \mathcal{F} h$, $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{G}} + \epsilon \boldsymbol{u}_{\mathrm{A}}$ and $b(\boldsymbol{x}) = 1 + \epsilon b_1(\boldsymbol{x})$ yields at order $\mathrm{O}(\epsilon)$ the QG equation for the dimensionless free surface height, see, e.g., [16],

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} \right) (\mathcal{F} h - \nabla^2 h + b_1 - f_1) = 0. \tag{2.53}$$

Thus, in the QG approximation, the potential vorticity, defined by

$$q = \mathcal{F} h - \nabla^2 h + b_1 - f_1 \tag{2.54}$$

is advected by the divergenceless geostrophic velocity $\boldsymbol{u}_{\mathrm{G}}$. The positive-definite operator $\mathcal{F} - \nabla^2$ is nondegenerate, so its operator inverse $1/(\mathcal{F} - \nabla^2)$ exists and is well defined on Fourier transformable functions, say. Therefore, the surface height h and its derivatives are determined uniquely from the potential vorticity q in QG theory.

The QG motion equation (2.53) also implies

$$\frac{\partial}{\partial t} \left(\frac{\mathcal{F} h^2}{2} + \frac{|\boldsymbol{\nabla} h|^2}{2} \right) = \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} h_{,t} - h \boldsymbol{u}_{\mathrm{G}} (\mathcal{F} h - \nabla^2 h + b_1 - f_1)). \tag{2.55}$$

Consequently, QG motion conserves the positive-definite energy;

$$E_{\mathrm{QG}} = \int \mathrm{d}x\, \mathrm{d}y \left(\tfrac{1}{2} \mathcal{F} h^2 + \tfrac{1}{2} |\boldsymbol{u}_{\mathrm{G}}|^2 \right), \tag{2.56}$$

provided the vector $\boldsymbol{\nabla} h$ in (2.55) is normal to the domain boundary (so $\boldsymbol{u}_{\mathrm{G}}$ is tangential there) and provided the boundary integral of the normal derivative of $h_{,t}$ vanishes [16].

Finally, the QG motion equation (2.53) yields the formal expression,

$$h_{,t} = \frac{1}{\mathcal{F} - \nabla^2}[\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla}(f_1 - b_1 + \nabla^2 h)], \tag{2.57}$$

whose gradient provides an estimate for the quantity $\partial_t(\hat{z} \times \boldsymbol{u}_{\mathrm{G}}) = -\boldsymbol{\nabla} h_{,t}$ appearing in expression (2.51) for $\boldsymbol{u}_{\mathrm{A}}$,

$$\boldsymbol{u}_{\mathrm{A}} = (\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \boldsymbol{u}_{\mathrm{G}} - \boldsymbol{\nabla} h_{,t} - f_1 \boldsymbol{u}_{\mathrm{G}}, \tag{2.58}$$

which may be written as

$$\boldsymbol{u}_{\mathrm{A}} = (\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \boldsymbol{u}_{\mathrm{G}} - \boldsymbol{\nabla} \frac{1}{\mathcal{F} - \nabla^2} J(h, \psi) - f_1 \boldsymbol{u}_{\mathrm{G}}, \tag{2.59}$$

where $\psi = f_1 - b_1 + \nabla^2 h$. Thus, in the QG approximation, the ageostrophic velocity may be expressed via (2.57) entirely in terms of the geostrophic velocity and other spatial derivatives of surface height.

In the following sections, we shall discuss constrained approximations of the RSW equations (2.38) which improve the QG approximation, while preserving the Hamiltonian structure of the unapproximated RSW equations. The Hamiltonian formulation of the QG approximation itself is discussed in e.g., [23].

3. Derivation of the model equations

3.1. Order O(1) and O(ϵ) model

In nearly geostrophic shallow water flow, the Rossby number is small $\epsilon \ll 1$ and the squared ratio of horizontal length scale to external Rossby deformation radius $\mathcal{F}$ is taken to be of order O(1). For simplicity in the following derivations of model equations, we will retain the assumptions $f(\boldsymbol{x}) = 1 + \epsilon f_1(\boldsymbol{x})$ and $b(\boldsymbol{x}) = 1 + \epsilon b_1(\boldsymbol{x})$ utilized in the QG approximation. These assumptions are not necessary, but are utilized because they allow the major points to be made, while simplifying the analysis. These assumptions also help clarify the nature of the expansion, since they imply that the well-known QG approximation will enter naturally as an intermediate step.

The fluid velocity may be represented as the sum of the leading order geostrophic velocity $\boldsymbol{u}_{\mathrm{G}}$ and a relatively smaller, order O(ϵ), ageostrophic velocity $\epsilon \boldsymbol{u}_{\mathrm{A}}$,

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{G}} + \epsilon \boldsymbol{u}_{\mathrm{A}} \tag{3.1}$$

with

$$\boldsymbol{u}_{\mathrm{G}} = \hat{z} \times \boldsymbol{\nabla}\left(\frac{\eta - b}{\epsilon \mathcal{F}}\right) = \hat{z} \times \boldsymbol{\nabla} h. \tag{3.2}$$

We substitute this representation of the fluid velocity into the shallow water HP, whose action is given in (2.42), to find *without* approximation,

$$L_{\mathrm{RSW}} = \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \left[\eta \boldsymbol{u} \cdot (\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}}) - \frac{(\eta - b)^2}{2\epsilon \mathcal{F}} - \frac{\epsilon}{2}\eta |\boldsymbol{u}_{\mathrm{G}} + \epsilon \boldsymbol{u}_{\mathrm{A}}|^2\right]. \tag{3.3}$$

As discussed earlier, we may now impose any approximation we choose for $\boldsymbol{u}_{\mathrm{A}}$ as a function of η and its derivatives. Moreover, the approximations for $\boldsymbol{u}_{\mathrm{A}}$ may be imposed independently in the momentum density $\delta L_{\mathrm{RSW}}/\delta \boldsymbol{u}$ and in the energy density. We may then use the theory presented in the previous section to cast this approximation into its constrained Hamiltonian form.

If $\epsilon \ll 1$, one may begin making approximations in (3.3) by simply dropping *both* $\boldsymbol{u}_{\mathrm{A}}$ terms of order $\mathrm{O}(\epsilon^2)$. The remaining terms of order O(1) and $\mathrm{O}(\epsilon)$ give

$$L_1 = \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \left[\eta \boldsymbol{u} \cdot (\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}}) - \frac{(\eta - b^2)}{2\epsilon \mathcal{F}} - \frac{\epsilon}{2} \eta |\boldsymbol{u}_{\mathrm{G}}|^2 \right], \tag{3.4}$$

which is in the form (2.28), with

$$\overline{\boldsymbol{m}}[\eta] = \frac{\delta L_1}{\delta \boldsymbol{u}} = \eta (\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}}), \qquad \overline{\mathcal{H}}[\eta] = \frac{(\eta - b)^2}{2\epsilon \mathcal{F}} + \frac{\epsilon}{2} \eta |\boldsymbol{u}_{\mathrm{G}}|^2. \tag{3.5}$$

The corresponding conserved Hamiltonian is, cf. (2.29),

$$H_1 = \int \mathrm{d}x\,\mathrm{d}y \left[(\boldsymbol{m} - \eta \boldsymbol{R} - \epsilon \eta \boldsymbol{u}_{\mathrm{G}}) \cdot \boldsymbol{u} + \frac{(\eta - b^2)}{2\epsilon \mathcal{F}} + \frac{\epsilon}{2} \eta |\boldsymbol{u}_{\mathrm{G}}|^2 \right]. \tag{3.6}$$

This Hamiltonian has variational derivatives given by

$$\begin{aligned} \delta H_1 = \int \mathrm{d}x\,\mathrm{d}y \Big\{ & \boldsymbol{u} \cdot \delta \boldsymbol{m} + (\boldsymbol{m} - \eta \boldsymbol{R} - \epsilon \eta \boldsymbol{u}_{\mathrm{G}}) \cdot \delta \boldsymbol{u} \\ & + \delta \eta \left[h + \frac{\epsilon}{2} |\boldsymbol{u}_{\mathrm{G}}|^2 + \frac{1}{\mathcal{F}} \hat{\boldsymbol{z}} \cdot \mathrm{curl} \eta (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}) - \boldsymbol{u} \cdot (\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}}) \right] \Big\} - \frac{1}{\mathcal{F}} \oint \mathrm{d}s \eta \delta \eta (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}) \cdot \hat{\boldsymbol{s}}, \end{aligned} \tag{3.7}$$

where $\delta h = \delta \eta / \epsilon \mathcal{F}$ and we have used the identity

$$\boldsymbol{a} \cdot \delta \boldsymbol{u}_{\mathrm{G}} = \mathrm{div}(\mathrm{a} \times \hat{\boldsymbol{z}} \delta h) - \delta h \hat{\boldsymbol{z}} \cdot \mathrm{curl} \boldsymbol{a}, \tag{3.8}$$

which holds for any vector function $\boldsymbol{a}$. In (3.7), $\hat{\boldsymbol{s}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}$ is the unit tangent vector on the boundary and $\hat{\boldsymbol{n}}$ is the unit outward normal vector. The boundary integral in (3.7) vanishes, provided the order $\mathrm{O}(\epsilon)$ ageostrophic velocity difference $\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}$ has no *tangential* component on the boundary, which we assume. The velocity $\boldsymbol{u}$ must also have no normal component on the boundary. Perhaps other boundary conditions could be imposed by using, say, the standard technique of adding a null Lagrangian to L_1. (A null Lagrangian is the space and time integral of a total divergence, whose only contribution in HP appears at the boundary. See, e.g., [7] for more discussion.) However, this approach is not pursued here.

The dynamics for $\overline{\boldsymbol{m}}/\eta$ at this order – $\mathrm{O}(\epsilon)$ – is given by Eqs. (2.32) and (2.33) as

$$\epsilon \frac{\partial \boldsymbol{u}_{\mathrm{G}}}{\partial t} + (\mathrm{curl}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}})) \times \boldsymbol{u} + \boldsymbol{\nabla} B_1 = 0, \qquad \frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot \eta \boldsymbol{u} = 0 \tag{3.9}$$

with Bernoulli function B_1 given by, cf. Eq. (2.25),

$$B_1 = \frac{\delta H_1}{\delta \eta} + \boldsymbol{u} \cdot \frac{1}{\eta} \overline{\boldsymbol{m}} = h + \frac{\epsilon}{2} |\boldsymbol{u}_{\mathrm{G}}|^2 + \frac{1}{\mathcal{F}} \hat{\boldsymbol{z}} \cdot \mathrm{curl}\, \eta (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}). \tag{3.10}$$

Note that the last term in Eq. (3.10) is of order $\mathrm{O}(\epsilon)$. For $\boldsymbol{\nabla} b = 0$, Eqs. (3.9) are equivalent to Salmon's model (with $f = 1$) derived in [19] by using HP with variations of particle paths at fixed time and at constant Lagrangian label (instead of variations of Lagrangian labels at constant Eulerian position, as done here). For $\boldsymbol{\nabla} b \neq 0$, Eqs. (3.9) are equivalent to those presented for Salmon's model with bottom topography in [2], where it was also shown that the boundary condition of zero tangential component of $\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}$ together with zero normal component of $\boldsymbol{u}$ ensures energy conservation.

The classical ideal fluid relations (2.23)–(2.26) all hold for the Eulerian version of Salmon's model given in Eqs. (3.9) and (3.10), since this model is derived from HP using a Lagrangian which – although it is linear in the velocity – still admits the symmetries of fluid particle relabeling and time translation. Thus, taking the vector product of the first equation in (3.9) with $\hat{z}$ gives the Bernoulli relation, cf. Eq. (2.23),

$$\eta Q_1 \boldsymbol{u} = \hat{z} \times \left(\boldsymbol{\nabla} B_1 + \epsilon \frac{\partial \boldsymbol{u}_{\mathrm{G}}}{\partial t} \right), \tag{3.11}$$

where Q_1 is the potential vorticity for this theory, cf. Eq. (2.36),

$$\eta Q_1 \equiv f + \hat{z} \cdot \mathrm{curl}\, \epsilon \boldsymbol{u}_{\mathrm{G}} = f + \epsilon \nabla^2 h, \tag{3.12}$$

in which $f = 1$ and ∇^2 is the horizontal Laplacian. As expected (cf. Eq. (2.26)) the potential vorticity Q_1 is advected by the flow

$$\frac{\partial Q_1}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} Q_1 = 0. \tag{3.13}$$

Rearranging (3.11) gives a modification of the QG relation (2.58),

$$\epsilon \boldsymbol{\nabla} h_{,t} = \hat{z} \times \boldsymbol{\nabla} B_1 - \eta Q_1 \boldsymbol{u}. \tag{3.14}$$

However, from the continuity equation for η we also have

$$\epsilon \mathcal{F} \boldsymbol{\nabla} h_{,t} = -\boldsymbol{\nabla} \mathrm{div}(\eta \boldsymbol{u}) \tag{3.15}$$

with $\eta = b + \epsilon \mathcal{F} h = 1 + \epsilon b_1 + \epsilon \mathcal{F} h$. Equating these two expressions for $\boldsymbol{\nabla} h_{,t}$ yields

$$\frac{1}{\mathcal{F}} \boldsymbol{\nabla} \mathrm{div}(\eta \boldsymbol{u}) + \hat{z} \times \boldsymbol{\nabla} \left(h + \frac{\epsilon}{2} |\boldsymbol{u}_{\mathrm{G}}|^2 + \frac{1}{\mathcal{F}} \hat{z} \cdot \mathrm{curl} \eta (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}) \right) - (f + \epsilon \boldsymbol{\nabla}^2 h) \boldsymbol{u} = 0, \tag{3.16}$$

which is a linear diagnostic partial differential equation for the fluid velocity (and Lagrange multiplier) $\boldsymbol{u}$, given the fluid depth η. In passing we note that Eq. (3.15) gives the following expression for the leading order contribution to $\boldsymbol{\nabla} h_{,t}$:

$$\mathcal{F} \boldsymbol{\nabla} h_{,t} = -\boldsymbol{\nabla} (\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla} b_1) - \boldsymbol{\nabla}\ \mathrm{div}\, \boldsymbol{u}_{\mathrm{A}} \tag{3.17}$$

with $\mathrm{div}\boldsymbol{u}_{\mathrm{A}}$ given in Eq. (2.52). The numerical solution of Salmon's HP model and its performance in comparison with solutions of the RSW equations are discussed in [2,5]. See also [18] for a recent discussion of Salmon's HP model.

3.2. The order $\mathrm{O}(\epsilon^2)$ *model*

Legendre transforming the unapproximated HP (3.3) for RSW dynamics with $\epsilon \boldsymbol{u}_{\mathrm{A}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}}$ leads to the following unconstrained Hamiltonian:

$$H_{\mathrm{RSW}} = \int \mathrm{d}x\, \mathrm{d}y \left[(\boldsymbol{m} - \eta(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}})) \cdot \boldsymbol{u} + \frac{1}{2\epsilon \mathcal{F}} (\eta - b)^2 + \frac{\epsilon^2}{2} \eta |\boldsymbol{u}_{\mathrm{G}} + \epsilon \boldsymbol{u}_{\mathrm{A}}|^2 \right]. \tag{3.18}$$

To obtain a class of order $\mathrm{O}(\epsilon^2)$ EG models, we impose a modeling ansatz for the ageostrophic velocity $\boldsymbol{u}_{\mathrm{A}}$, based on the form of the QG result (2.58) for $\boldsymbol{u}_{\mathrm{A}}$. Thus, we choose

$$H_2 = \int \mathrm{d}x\, \mathrm{d}y \left[(\boldsymbol{m} - \eta(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}})) \cdot \boldsymbol{u} + \frac{1}{2\epsilon \mathcal{F}} (\eta - b)^2 + \frac{\epsilon}{2} \eta |\boldsymbol{u}_{\mathrm{G}} + \gamma \epsilon \boldsymbol{u}_{\mathrm{A}}|^2 \right], \tag{3.19}$$

where $\boldsymbol{u}_{\rm G} = \hat{z} \times \nabla h$ and $\boldsymbol{u}_{\rm A}$ is the prescribed function

$$\boldsymbol{u}_{\rm A} = \tau(\boldsymbol{u}_{\rm G} \cdot \nabla)\hat{z} \times \boldsymbol{u}_{\rm G} - \alpha\tau\nabla\left(\frac{1}{\mathcal{F} - \nabla^2} J(h, \psi)\right) - \beta\tau f_1 \boldsymbol{u}_{\rm G} \tag{3.20}$$

and $\psi = f_1 - b_1 + \nabla^2 h$. The constants τ, α, β, and γ are regarded as free parameters. As we shall show, imposing this ansatz for $\boldsymbol{u}_{\rm A}$ in the constrained Hamiltonian H_2 leads to a parameterized family of EG models. This family includes the well-known GM model. These EG models will conserve the positive EG energy

$$E_2 = \int \mathrm{d}x\,\mathrm{d}y\left(\frac{(\eta - b)^2}{2\epsilon\mathcal{F}} + \frac{\epsilon}{2}\eta|\boldsymbol{u}_{\rm G} + \gamma\epsilon\boldsymbol{u}_{\rm A}|^2\right), \tag{3.21}$$

and will conserve the EG potential vorticity

$$Q_2 = \frac{1}{\eta}\hat{z} \cdot \mathrm{curl}(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\rm G} + \epsilon^2\boldsymbol{u}_{\rm A}), \tag{3.22}$$

on fluid particles.

3.2.1. The Geostrophic momentum model

Before substituting the modeling choice (3.20) for $\boldsymbol{u}_{\rm A}$ into the constrained Hamiltonian H_2 in (3.18) and taking variations, we briefly review the properties of the GM model. The GM model is described by Hoskins [14,15] (See also [8], which is referenced in [14].) Recent developments and extensions of the GM model are discussed by Cullen et al. [6], Roulstone and Norbury [17] and Roulstone and Sewell [18]. We review the GM model for the case in which variations of the Coriolis parameter with position are ignored, i.e., $f_1 = 0$. The momentum is attributed to the geostrophic motion, so that

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{\rm G} + \hat{z} \times \boldsymbol{u} + \nabla h = 0, \qquad \frac{\partial\eta}{\partial t} + \nabla \cdot \eta\boldsymbol{u} = 0 \tag{3.23}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla, \qquad \boldsymbol{u}_{\rm G} = \hat{z} \times \nabla h, \qquad h = \frac{\eta - b(\boldsymbol{x})}{\epsilon\mathcal{F}}. \tag{3.24}$$

The motion equation in (3.23) gives two linear equations for the two components of the velocity which we may solve to find

$$(1 + \epsilon\zeta_{\rm GM})\boldsymbol{u} = \hat{z} \times (\nabla B_{\rm G} + \epsilon\boldsymbol{u}_{{\rm G},t}) + \epsilon^2 J(h_{,t}, \boldsymbol{u}_{\rm G}), \tag{3.25}$$

in which

$$B_{\rm G} = h + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\rm G}|^2, \qquad \zeta_{\rm GM} = \nabla^2 h + \epsilon J(u_{\rm G}, v_{\rm G}) = \hat{z} \cdot \mathrm{curl}[\boldsymbol{u}_{\rm G} - \tfrac{1}{2}\epsilon(\boldsymbol{u}_{\rm G} \cdot \nabla)\hat{z} \times \boldsymbol{u}_{\rm G}], \tag{3.26}$$

and J denotes the Jacobian, e.g., $J(a, b) = a_{,x}b_{,y} - b_{,x}a_{,y}$ for any two functions, a and b. In fact, the quantity

$$J(u_{\rm G}, v_{\rm G}) = \frac{\partial u_{\rm G}}{\partial x}\frac{\partial v_{\rm G}}{\partial y} - \frac{\partial v_{\rm G}}{\partial x}\frac{\partial u_{\rm G}}{\partial y} = \det(h_{,ij}) \tag{3.27}$$

is the Hessian determinant and equilibrium GM solutions satisfy

$$J\left(\frac{1 + \epsilon\nabla^2 h + \epsilon^2\det(h_{,ij})}{1 + \epsilon\mathcal{F}h}, h + \epsilon|\nabla h|^2\right)\Bigg|_{\rm equilibrium} = 0, \tag{3.28}$$

which is a geometrically intriguing generalization of the QG condition, $J(q, h) = 0$ at equilibrium, with q defined in (2.54).

Rearranging (3.25) expresses the GM motion equation as

$$\epsilon \boldsymbol{u}_{G,t} + (1 + \epsilon\zeta_{GM})\hat{z} \times \boldsymbol{u} + \nabla B_G + \epsilon^2 J(h_{,t}, \nabla h) = 0, \tag{3.29}$$

in which we note the relations

$$J(h_{,t}, \nabla h) = (\boldsymbol{u}_{G,t} \cdot \nabla)\nabla h = -(\boldsymbol{u}_{G,t} \cdot \nabla)\hat{z} \times \boldsymbol{u}_G. \tag{3.30}$$

Expanding $\nabla J(h_{,t}, h)$ by the chain rule and using the definition of the geostrophic velocity $\boldsymbol{u}_G = \hat{z} \times \nabla h$ gives

$$\hat{z} \times \nabla J(h_{,t}, h) = (\boldsymbol{u}_{G,t} \cdot \nabla)\boldsymbol{u}_G - (\boldsymbol{u}_G \cdot \nabla)\boldsymbol{u}_{G,t}, \tag{3.31}$$

since $J(h_{,t}, h) = \hat{z} \cdot \boldsymbol{u}_{G,t} \times \boldsymbol{u}_G$. Consequently, we have the identity

$$\begin{aligned}\frac{\partial}{\partial t}((\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G &= 2(\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_{G,t} - \nabla J(h_{,t}, h) = 2(\boldsymbol{u}_{G,t} \cdot \nabla)\hat{z} \times \boldsymbol{u}_G + \nabla J(h_{,t}, h)\\ &= -2J(h_{,t}, \nabla h) + \nabla J(h_{,t}, h),\end{aligned} \tag{3.32}$$

and we find that

$$J(h_{,t}, \nabla h) = -\frac{1}{2}\frac{\partial}{\partial t}((\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G) + \tfrac{1}{2}\nabla J(h_{,t}, h), \tag{3.33}$$

or, equivalently,

$$-(\boldsymbol{u}_{G,t} \cdot \nabla)\hat{z} \times \boldsymbol{u}_G = -\frac{1}{2}\frac{\partial}{\partial t}((\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G) + \frac{1}{2}\nabla(\hat{z} \cdot \boldsymbol{u}_{G,t} \times \boldsymbol{u}_G). \tag{3.34}$$

Therefore, we may write the GM motion equation in (3.23), or equivalently, Eq. (3.29) in a form which will be useful later, as

$$\begin{aligned}&\frac{\partial}{\partial t}(\epsilon \boldsymbol{u}_G - \tfrac{1}{2}\epsilon^2(\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G) + (\mathrm{curl}(\boldsymbol{R}_0 + \epsilon \boldsymbol{u}_G - \tfrac{1}{2}\epsilon^2(\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G)) \times \boldsymbol{u}\\ &\quad + \nabla(B_G + \tfrac{1}{2}\epsilon^2 J(h_{,t}, h)) = 0,\end{aligned} \tag{3.35}$$

where $\boldsymbol{R}_0 = \frac{1}{2}(-y, x)$, so that curl $\boldsymbol{R}_0 = \hat{z}$. This equation is in the same form as the constrained Hamiltonian motion equation (2.32) with Eulerian momentum density defined by

$$\frac{1}{\eta}\overline{\boldsymbol{m}} = \boldsymbol{R}_0 + \epsilon \boldsymbol{u}_G - \frac{\epsilon^2}{2}(\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G. \tag{3.36}$$

The factor $-\frac{1}{2}$ in the ageostrophic velocity $\boldsymbol{u}_A = -\frac{1}{2}(\boldsymbol{u}_G \cdot \nabla)\hat{z} \times \boldsymbol{u}_G$ here, relative to the corresponding factor of $+1$ for QG in (2.58), is a peculiarity of the GM model. Nonetheless, the classical ideal fluid relations (2.18)–(2.26) and conservation of the EG energy (3.21) (with $\gamma = 0$) all hold for the GM model at order $O(\epsilon^2)$. For example, the continuity equation for η and the curl of the motion equation (3.35) imply potential vorticity advection for GM, $\mathrm{d}\overline{Q}_{GM}/\mathrm{d}t = 0$, with $\eta\overline{Q}_{GM} = 1 + \epsilon\zeta_{GM}$. Hence, the geometrically intriguing condition (3.28) for GM equilibria is expressible in the briefer standard form, $J(\overline{Q}_{GM}, B_G) = 0$ at equilibrium, which stems from preservation in the GM model of the underlying Lie–Poisson Hamiltonian structure (2.16) for RSW. (See, e.g., [9,13] for discussions of the relation between fluid equilibrium conditions and Lie–Poisson Hamiltonian structure.) In the following sections, we shall place the GM model into a family of EG models derived from HP by modeling RSW using Lagrangians that are linear in the fluid velocity. The classical ideal fluid relations satisfied by the GM model then follow as consequences of HP.

3.2.2. Variational calculation at order $\mathrm{O}(\epsilon^2)$

Our constrained Hamiltonian is given by (3.19) where we use the modeling ansatz (3.20) for $\boldsymbol{u}_{\mathrm{A}}$. Taking variations gives

$$\delta H_2 = \int \mathrm{d}x\,\mathrm{d}y\big\{\boldsymbol{u}\cdot\delta\boldsymbol{m} + (\boldsymbol{m} - \eta\boldsymbol{R} - \epsilon\eta\boldsymbol{u}_{\mathrm{G}} - \epsilon^2\eta\boldsymbol{u}_{\mathrm{A}})\cdot\delta\boldsymbol{u} + \delta\eta\big[h + \tfrac{1}{2}\epsilon|\boldsymbol{u}_{\mathrm{G}} + \gamma\epsilon\boldsymbol{u}_{\mathrm{A}}|^2$$
$$- \boldsymbol{u}\cdot(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{G}} + \epsilon^2\boldsymbol{u}_{\mathrm{A}})\big] - \epsilon\eta\overline{\boldsymbol{u}}^{(\gamma)}\cdot\delta\boldsymbol{u}_{\mathrm{G}} - \epsilon^2\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot\delta\boldsymbol{u}_{\mathrm{A}}\big\}, \tag{3.37}$$

where we denote

$$\overline{\boldsymbol{u}}^{(\gamma)} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}} - \gamma\epsilon\boldsymbol{u}_{\mathrm{A}}, \tag{3.38}$$
$$\hat{\boldsymbol{u}}^{(\gamma)} = \boldsymbol{u} - \gamma\boldsymbol{u}_{\mathrm{G}} - \gamma^2\epsilon\boldsymbol{u}_{\mathrm{A}} \tag{3.39}$$

and $\boldsymbol{u}_{\mathrm{A}}$ is given in Eq. (3.20) in terms of h. The next to last term in (3.37) is given by

$$\delta_0 H_2 \equiv -\epsilon\int \mathrm{d}x\,\mathrm{d}y\eta\overline{\boldsymbol{u}}^{(\gamma)}\cdot\delta\boldsymbol{u}_{\mathrm{G}} = \epsilon\int \mathrm{d}x\,\mathrm{d}y\delta h\hat{\boldsymbol{z}}\cdot\mathrm{curl}\eta\overline{\boldsymbol{u}}^{(\gamma)} - \epsilon\oint \mathrm{d}s\delta h\eta\hat{\boldsymbol{s}}\cdot\overline{\boldsymbol{u}}^{(\gamma)}. \tag{3.40}$$

We separate the last term in (3.37) into three pieces by setting

$$-\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot\delta\boldsymbol{u}_{\mathrm{A}} = \delta H_{\mathrm{rot}} + \delta_1 H_2 + \delta_2 H_2, \tag{3.41}$$

where

$$\delta H_{\mathrm{rot}} = \beta\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot f_1(\boldsymbol{x})\delta\boldsymbol{u}_{\mathrm{G}}, \tag{3.42}$$
$$\delta_1 H_2 = -\alpha\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot\boldsymbol{\nabla}\left(\frac{-1}{\mathcal{F} - \nabla^2}\right)[J(\delta h, \psi) + J(h, \nabla^2\delta h)], \tag{3.43}$$
$$\delta_2 H_2 = -\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot[(\delta\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{z}}\times\boldsymbol{u}_{\mathrm{G}} + (\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{z}}\times\delta\boldsymbol{u}_{\mathrm{G}}]. \tag{3.44}$$

Beginning with δH_{rot}, we utilize (3.8) to find

$$\delta H_{\mathrm{rot}} = -\beta\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y[\delta h\hat{\boldsymbol{z}}\cdot\mathrm{curl}(\eta f_1(\boldsymbol{x})\hat{\boldsymbol{u}}^{(\gamma)})] + \frac{\beta\tau\epsilon^2}{2}\oint \mathrm{d}s[\eta\delta h f_1\hat{\boldsymbol{u}}^{(\gamma)}\cdot\hat{\boldsymbol{s}}]. \tag{3.45}$$

Next we compute $\delta_1 H_2$,

$$\delta_1 H_2 = -\alpha\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\eta\hat{\boldsymbol{u}}^{(\gamma)}\cdot\boldsymbol{\nabla}\frac{-1}{\mathcal{F} - \nabla^2}[\delta\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla}\nabla^2\delta h]$$
$$= -\alpha\tau\epsilon^2\int \mathrm{d}x\,\mathrm{d}y\delta h\left\{\hat{\boldsymbol{z}}\cdot\mathrm{curl}\left[\left(\frac{1}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\right)\boldsymbol{\nabla}\psi\right] + \nabla^2\left(J\left(h, \frac{1}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\right)\right)\right\}$$
$$- \alpha\tau\epsilon^2\oint \mathrm{d}s\hat{\boldsymbol{n}}\cdot\eta\hat{\boldsymbol{u}}^{(\gamma)}\left[\frac{-1}{\mathcal{F} - \nabla^2}(\delta\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla}\nabla^2\delta h)\right]$$
$$- \alpha\tau\epsilon^2\oint \mathrm{d}s\left(\frac{-1}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\right)[\hat{\boldsymbol{n}}\cdot\boldsymbol{u}_{\mathrm{G}}(\nabla^2\delta h) + \delta h\hat{\boldsymbol{s}}\cdot\boldsymbol{\nabla}\psi], \tag{3.46}$$

where $\delta h = \delta\eta/\epsilon\mathcal{F}$, and $\hat{\eta}_{,t}$ is defined by

$$\hat{\eta}_{,t} = -\mathrm{div}\eta\hat{\boldsymbol{u}}^{(\gamma)} = \eta_t + \gamma\mathrm{div}\eta(\boldsymbol{u}_{\mathrm{G}} + \gamma\epsilon\boldsymbol{u}_{\mathrm{A}}), \tag{3.47}$$

and we have integrated by parts several times using the identity (3.8). Similarly, we compute the other variation, (3.44), as

$$\begin{aligned}\delta_2 H_2 &= -\tau\epsilon^2 \int \mathrm{d}x\,\mathrm{d}y \eta \hat{\boldsymbol{u}}^{(\gamma)} \cdot [(\delta \boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \boldsymbol{u}_{\mathrm{G}} + (\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \delta \boldsymbol{u}_{\mathrm{G}}] \\ &= -\tau\epsilon^2 \int \mathrm{d}x\,\mathrm{d}y \delta h \hat{z} \cdot \mathrm{curl}[2\eta(\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} h + (\mathrm{div}\eta\hat{\boldsymbol{u}}^{(\gamma)})\boldsymbol{\nabla} h] \\ &\quad + \tau\epsilon^2 \oint \mathrm{d}s\, \hat{\boldsymbol{n}} \cdot [\boldsymbol{u}_{\mathrm{G}}(\eta\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla}\delta h) - \delta h((\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla}\eta\hat{\boldsymbol{u}}^{(\gamma)} + (\eta\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla}\boldsymbol{u}_{\mathrm{G}})]. \end{aligned} \tag{3.48}$$

For simplicity, we may set the boundary integral terms in (3.40), (3.45), (3.46) and (3.48) equal to zero by assuming the domain is infinite with $\boldsymbol{u}$ and $\boldsymbol{\nabla} h$ vanishing at infinity. We also assume there is no difficulty with boundary conditions in taking $1/(\mathcal{F} - \nabla^2)$ to be a symmetric operator when integrating by parts.

After completing this variational computation, we find

$$\begin{aligned}&\delta_0 H_2 + \delta H_{\mathrm{rot}} + \delta_1 H_2 + \delta_2 H_2 \\ &= \epsilon \int \mathrm{d}x\,\mathrm{d}y \delta h \hat{z} \cdot \mathrm{curl}\eta\overline{\boldsymbol{u}}^{(\gamma)} - \epsilon^2 \int \mathrm{d}x\,\mathrm{d}y \Big\{ \tau\delta h \hat{z} \cdot \mathrm{curl}\Big[2\eta(\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} h - \hat{\eta}_{,t}\boldsymbol{\nabla} h \\ &+ \Big(\frac{\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big)\boldsymbol{\nabla}\psi + \beta\eta f_1 \hat{\boldsymbol{u}}^{(\gamma)}\Big] + \tau\delta h \nabla^2 J\Big(h, \frac{\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big)\Big\}. \end{aligned} \tag{3.49}$$

Hence, with $\epsilon\delta h = \delta\eta/\mathcal{F}$,

$$\begin{aligned}\delta H_2 = \int \mathrm{d}x\,\mathrm{d}y \Big\{ &\boldsymbol{u} \cdot \delta \boldsymbol{m} + (\boldsymbol{m} - \eta \boldsymbol{R} - \epsilon\eta \boldsymbol{u}_{\mathrm{G}} - \epsilon^2 \eta \boldsymbol{u}_{\mathrm{A}}) \cdot \delta \boldsymbol{u} \\ &+ \delta\eta \Big[h + \frac{\epsilon}{2}|\boldsymbol{u}_{\mathrm{G}} + \gamma\epsilon \boldsymbol{u}_{\mathrm{A}}|^2 - \boldsymbol{u} \cdot (\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}}) \Big] \\ &+ \delta\eta \frac{1}{\mathcal{F}}\hat{z} \cdot \mathrm{curl}\Big[\eta(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}} - \gamma\epsilon \boldsymbol{u}_{\mathrm{A}}) - \epsilon\tau 2\eta(\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} h \\ &+ \epsilon\tau\hat{\eta}_{,t}\boldsymbol{\nabla} h - \epsilon\tau\beta\eta f_1 \hat{\boldsymbol{u}}^{(\gamma)} + \psi\boldsymbol{\nabla}\Big(\frac{\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big)\Big] + \delta\eta\frac{\epsilon}{\mathcal{F}}\tau\nabla^2 J\Big(h, \frac{-\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big)\Big\}. \end{aligned} \tag{3.50}$$

The equation of motion which results from inserting these variational derivatives into (2.32) is

$$\frac{\partial}{\partial t}(\epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}}) + \mathrm{curl}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} + \epsilon^2 \boldsymbol{u}_{\mathrm{A}}) \times \boldsymbol{u} + \boldsymbol{\nabla} B_2 = 0, \tag{3.51}$$

in which the Bernoulli function is expressed as, cf. Eq. (2.25),

$$\begin{aligned}B_2 = h + \frac{\epsilon}{2}|\boldsymbol{u}_{\mathrm{G}} + \gamma\epsilon \boldsymbol{u}_{\mathrm{A}}|^2 + \frac{1}{\mathcal{F}}\hat{z} \cdot \mathrm{curl}\Big[&\eta(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}} - \gamma\epsilon \boldsymbol{u}_{\mathrm{A}}) - \epsilon\tau 2\eta(\hat{\boldsymbol{u}}^{(\gamma)} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} h \\ &+ \epsilon\tau\hat{\eta}_{,t}\boldsymbol{\nabla} h - \epsilon\tau\beta\eta f_1 \hat{\boldsymbol{u}}^{(\gamma)} + \psi\boldsymbol{\nabla}\Big(\frac{\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big)\Big] + \frac{\epsilon\tau}{\mathcal{F}}\nabla^2 J\Big(h, \frac{-\alpha}{\mathcal{F} - \nabla^2}\hat{\eta}_{,t}\Big), \end{aligned} \tag{3.52}$$

where $\hat{\eta}_{,t} = -\mathrm{div}\eta\hat{\boldsymbol{u}}^{(\gamma)}$ and the ageostrophic velocity $\boldsymbol{u}_{\mathrm{A}}$ is given by (3.20).

For $\tau = -\frac{1}{2}$, $\alpha = \beta = \gamma = 0$, and $f_1 = b_1 = 0$, the motion equation (3.51) reduces to

$$\begin{aligned}&\frac{\partial}{\partial t}(\epsilon \boldsymbol{u}_{\mathrm{G}} - \tfrac{1}{2}\epsilon^2(\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \boldsymbol{u}_{\mathrm{G}}) + (\mathrm{curl}(\boldsymbol{R} + \epsilon \boldsymbol{u}_{\mathrm{G}} - \tfrac{1}{2}\epsilon^2(\boldsymbol{u}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\hat{z} \times \boldsymbol{u}_{\mathrm{G}}) \times \boldsymbol{u}) \\ &\quad + \boldsymbol{\nabla}\Big\{ h + \frac{\epsilon}{2}|\boldsymbol{u}_{\mathrm{G}}|^2 - \frac{\epsilon}{2\mathcal{F}}\hat{z} \cdot \mathrm{curl}[h\boldsymbol{\nabla}\eta_{,t}] + \frac{1}{\mathcal{F}}\hat{z} \cdot \mathrm{curl}\Big[\eta\Big(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}} + \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nabla} h\Big)\Big]\Big\} = 0. \end{aligned} \tag{3.53}$$

Eq. (3.53) is equivalent to the GM equation (3.35) with $b_1 = 0$. In this case, $\eta = 1 + \epsilon \mathcal{F} h$ and the last term of B_2 in Eq. (3.53) reduces to $\eta\hat{z}\times$ the left-hand side of the GM motion equation in (3.23). Thus in this case the Hamiltonian EG model equation (3.53) is expressible as a result of Eq. (3.35) in the form

$$\boldsymbol{GM} - \frac{1}{\mathcal{F}}\boldsymbol{\nabla}\hat{z}\cdot \mathrm{curl}(\eta\hat{z}\times \boldsymbol{GM}) = \left(\mathcal{I} - \frac{1}{\mathcal{F}}\boldsymbol{\nabla}\mathrm{div}\eta\right)\boldsymbol{GM} = 0. \tag{3.54}$$

where $\boldsymbol{GM}$ is the left-hand side of the GM motion equation in (3.23), or equivalently, the left-hand side of Eq. (3.35), and $\mathcal{I}$ is the identity matrix. The GM equation (3.35) for $b_1 \neq 0$ is recovered with the choice

$$\boldsymbol{u}_{\mathrm{A}} = -\frac{1}{2}(\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla})\hat{z}\times \boldsymbol{u}_{\mathrm{G}} - \frac{1}{2\mathcal{F}}\boldsymbol{\nabla}J(h, b_1). \tag{3.55}$$

Note that last term in (3.55) is equivalent to a part of the second term in (3.20) with $\tau = -\frac{1}{2}$ and $\alpha = 1$.

For the modeling parameters set equal to values indicated by the QG result (2.59), i.e., for $\tau = \alpha = \beta = 1$, and with the choice $\gamma = 1$ so that the conserved kinetic energy density in E_2 is $\frac{1}{2}\epsilon\eta|\boldsymbol{u}_{\mathrm{G}} + \epsilon\boldsymbol{u}_{\mathrm{A}}|^2$, the Bernoulli function in (3.52) is

$$\begin{aligned} B_2 = h + \frac{\epsilon}{2}|\boldsymbol{u}_{\mathrm{G}} + \epsilon\boldsymbol{u}_{\mathrm{A}}|^2 + \frac{1}{\mathcal{F}}\hat{z}\cdot \mathrm{curl}\Bigg[&\eta(\hat{\boldsymbol{u}} - \epsilon 2(\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}h - \epsilon f_1\hat{\boldsymbol{u}}) \\ &+ \epsilon\hat{\eta}_{,t}\boldsymbol{\nabla}h + \psi\boldsymbol{\nabla}\left(\frac{1}{\mathcal{F}-\nabla^2}\hat{\eta}_{,t}\right)\Bigg] + \frac{\epsilon}{\mathcal{F}}\nabla^2 J\left(h, \frac{-1}{\mathcal{F}-\nabla^2}\hat{\eta}_{,t}\right), \end{aligned} \tag{3.56}$$

where we denote

$$\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{G}} - \epsilon\boldsymbol{u}_{\mathrm{A}} \quad \text{and} \quad \hat{\eta}_{,t} = -\mathrm{div}\eta\hat{\boldsymbol{u}}. \tag{3.57}$$

When $\hat{\boldsymbol{u}} = 0$, Eqs. (3.51) and (3.56) recover the RSW motion equation in (2.47) with small variations in Coriolis parameter and bottom topography.

3.3. Summary comments

By construction, each member of the family of Hamiltonian EG model equations (3.51) – for any choice of $(\tau, \alpha, \gamma, \beta)$ and for any functional forms of $\mathrm{curl}\,\boldsymbol{R}(\boldsymbol{x})$, $f_1(\boldsymbol{x})$ and $b_1(\boldsymbol{x})$ – possesses a Kelvin circulation theorem, cf. Eq. (2.21), in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{G}} + \epsilon^2\boldsymbol{u}_{\mathrm{A}})\cdot \mathbf{d}\boldsymbol{x} = 0. \tag{3.58}$$

These equations therefore admit an advection law for the potential vorticity Q_2,

$$\frac{\partial Q_2}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}Q_2 = 0 \quad \text{with } Q_2 = \frac{1}{\eta}\hat{z}\cdot \mathrm{curl}(\boldsymbol{R} + \epsilon\boldsymbol{u}_{\mathrm{G}} + \epsilon^2\boldsymbol{u}_{\mathrm{A}}), \tag{3.59}$$

in which $\boldsymbol{u}_{\mathrm{A}}$ is given in Eq. (3.20). The Hamiltonian EG equations (3.51) also satisfy the Bernoulli relation,

$$\eta Q_2\boldsymbol{u} = \hat{z}\times\left(\boldsymbol{\nabla}B_2 + \frac{\partial}{\partial t}(\epsilon\boldsymbol{u}_{\mathrm{G}} + \epsilon^2\boldsymbol{u}_{\mathrm{A}})\right), \tag{3.60}$$

with B_2 given in (3.52). The Bernoulli relation (3.60) is one point of departure for an iterative solution for $\boldsymbol{u}$. Rearranging (3.60) and using the continuity equation for η gives

$$\epsilon\boldsymbol{\nabla}h_{,t} = \frac{-1}{\mathcal{F}}\boldsymbol{\nabla}\mathrm{div}\eta\boldsymbol{u} = \hat{z}\times\boldsymbol{\nabla}B_2 - \eta Q_2\boldsymbol{u} + \epsilon^2\hat{z}\times\frac{\partial\boldsymbol{u}_{\mathrm{A}}}{\partial t}. \tag{3.61}$$

which is a linear diagnostic partial differential equation for the fluid velocity (and Lagrange multiplier) $\boldsymbol{u}$, given the fluid depth η. This equation may be approached in various ways, e.g., by first dropping order $O(\epsilon^2)$ terms, then iterating.

The family of equations (3.51) conserves the EG energy E_2 in (3.21) and the Casimir functions C_Φ in (2.37). Therefore, this family of models admits Hamiltonian methods for identifying classes of steady solutions as relative equilibria (critical points of the sum $H_\Phi = E_2 + C_\Phi$) and studying their stability properties using the energy-Casimir method, discussed, e.g., in [13] and references therein.

Note that by choosing the functional form of $\boldsymbol{u}_A$ and the values for the modeling parameters $(\tau, \alpha, \gamma, \beta)$, the particular forms of the potential vorticity Q_2 and the energy E_2 that are conserved can be *specified*. The choice $\tau = \alpha = \beta = 1$, so that $\boldsymbol{u}_A$ agrees with (2.59), which is given by an asymptotic expansion of the RSW equations to $O(\epsilon)$, and the choice of $\gamma = 1$, which results in kinetic energy conservation involving $|\boldsymbol{u}_G + \epsilon\boldsymbol{u}_A|^2$, give unified boundary conditions and expressions for Q_2 and E_2 with apparently highest order dynamical consistency. Thus, this particular new model, which is given by the motion equation (3.51) with Bernoulli function (3.56), may be expected to give more accurate approximate solutions than either GM or Salmon's model does. We summarize the equations of motion, Bernoulli function and ageostrophic velocity for this model as follows:

$$\frac{\partial}{\partial t}(\epsilon\boldsymbol{u}_G + \epsilon^2\boldsymbol{u}_A) - \boldsymbol{u} \times \mathrm{curl}(\boldsymbol{R} + \epsilon\boldsymbol{u}_G + \epsilon^2\boldsymbol{u}_A) + \nabla B_2 = 0, \tag{3.62}$$

$$\begin{aligned} B_2 = h + \frac{\epsilon}{2}|\boldsymbol{u}_G + \epsilon\boldsymbol{u}_A|^2 + \frac{1}{\mathcal{F}}\hat{z}\cdot\mathrm{curl}\Big[\eta(\hat{\boldsymbol{u}} - 2\epsilon(\hat{\boldsymbol{u}}\cdot\nabla)\nabla h - \epsilon f_1\hat{\boldsymbol{u}}) \\ + \epsilon\hat{\eta}_{,t}\nabla h + \psi\nabla\left(\frac{1}{\mathcal{F}-\nabla^2}\hat{\eta}_{,t}\right)\Big] + \frac{\epsilon}{\mathcal{F}}\nabla^2 J\left(h, \frac{-1}{\mathcal{F}-\nabla^2}\hat{\eta}_{,t}\right), \end{aligned} \tag{3.63}$$

$$\boldsymbol{u}_A = (\boldsymbol{u}_G\cdot\nabla)\hat{z}\times\boldsymbol{u}_G - \nabla\left(\frac{1}{\mathcal{F}-\nabla^2}J(h,\psi)\right) - f_1\boldsymbol{u}_G, \tag{3.64}$$

where $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_G - \epsilon\boldsymbol{u}_A$ and $\hat{\eta}_{,t} = -\mathrm{div}\,\eta\hat{\boldsymbol{u}}$. Both the stability characteristics of equilibrium solutions of this family of Hamiltomian EG models of RSW motion and their solution by numerical integration will be discussed elsewhere. Finally, we point out that these results can be formally generalized to account for $O(1)$ variations in bottom topography $b(\boldsymbol{x})$, i.e., for $\nabla b = O(1)$, and to include a Coriolis parameter $f(\boldsymbol{x})$ with general spatial variability.

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